

Non-differentiability of the effective potential and the replica symmetry breaking in the random energy model

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Abstract. The effective potential for the two-replica system of the random energy model is exactly derived. It is an analytic function of the magnetizations of two replicas, φ^1 and φ^2 in the high-temperature phase. In the low-temperature phase, where the replica symmetry breaking takes place, the effective potential becomes non-analytic when $\varphi^1 = \varphi^2$. The non-analyticity is considered as a consequence of the condensation of the Boltzmann measure, which is a typical property of a glass phase.

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1. Introduction

A technical difficulty in theoretical study of quenched disordered systems originates from inhomogeneity due to disordered environment. In those systems, we first take the thermal average of physical quantities in a fixed disordered environment and then take the average over the disorder. However, if we can first average out the disorder, the systems become homogeneous and problems will be more tractable. Several methods to make it possible were developed in the last four decades.

One of the standard method will be the replica trick [1–3]. Namely, a partition function of identical n copies (replicas) of a disordered system is introduced and then the average over the disorder is taken. The resultant partition function defines a homogeneous “replicated” system. According to the replica trick, in order to extract disorder-averaged physical quantities from the replicated system, the zero-replica limit $n \rightarrow 0$ is taken despite that n is a positive integer. Although there are several studies for exact replica approach to specific models [4–7], general mathematical foundation has not been found yet [8, 9].

In mean-field models such as the Sherrington-Kirkpatrick model [10] or the random energy model (REM) [11, 12], glassy behaviour comes out together with the replica symmetry breaking (RSB). The RSB originally means that the symmetry under permutation of the replica indices is (spontaneously) broken in a replicated system. It is brought about by dominance of saddle points that break the replica symmetry when evaluating the partition function of a replicated system. Since the evaluation is carried out in the zero-replica limit $n \rightarrow 0$, the original definition of the RSB is mathematically ambiguous. However, physical insights clarify that the RSB is a consequence of contribution from metastable states, which can be measured by the probability distribution of the two-replica overlap. Thus a well-defined order parameter of the RSB is extracted from the probability distribution, which is referred to as the Parisi order parameter [13, 14].

As for short-ranged models, Le Doussal and Wiese showed, in study of random elastic models, that the RSB and non-analyticity of the effective potential in the replicated system appear at the same time when the system goes into a glass phase from the high-temperature phase [15, 16]. If this phenomenon is confirmed in various quenched random systems, the non-analyticity in effective potential may be regarded as an indication of the RSB. For this reason, it is worthwhile to examine universality of relationship between the non-analytic effective potential and the RSB.

In this paper, we compute the effective potential for the replicated system consisting of the REM and attempt better understanding of the relationship. The model is simple, so that we can exactly calculate the effective potential without use of the replica trick. Hence, we can examine analyticity of the effective potential without suffering from artifact by approximation and from mathematical ambiguity caused by the replica trick.

This paper is organized as follows: in the next section, we introduce two definitions of the effective potential: one is defined from the Legendre transform of the cumulant

generating function, which is adopted by the literature [15–21]. The other is so-called the constraint effective potential [22, 23], which is defined as the free energy with an order parameter fixed. The relationship of the two effective potentials is known, which is also described [23, 24]. In section 3, we introduce the REM and compute the generating function of the replicated system with two replicas. In section 4, the effective potential is derived by the Legendre transform of the generating function. The constraint effective potential is also computed in section 5. We discuss the origin of non-analyticity of the effective potential in the last section.

2. The effective potential in a replicated system

In this section, we first recall the effective potential in a replicated system introduced in [15–21] with a little modification along the present work. Next, we introduce the constraint effective potential [22, 23] in a replicated system.

Consider a field theory on a lattice described by a Hamiltonian $H_{\text{DO}}[u]$. Here $u := \{u_i\}_i$ denotes a field variable with the site index $i \in \{1, \dots, N\}$. Note that the Hamiltonian depends on not only u but also disordered environment. Suppose that u is coupled to a uniform external source h . The theory in the inverse temperature β is described by the partition function

$$Z(h) := \int \mathcal{D}u \, e^{\beta(-H_{\text{DO}}[u] + Nh\tilde{u})},$$

where

$$\tilde{u} := \frac{1}{N} \sum_i u_i.$$

When u is a spin variable, \tilde{u} corresponds with the magnetization per site. The “thermal” cumulants of \tilde{u} (i.e., cumulants of \tilde{u} with respect to the thermal average) at $h = 0$ with fixed disorder can be obtained from the series for $\log Z(h)$ as a function of h . Thus the disorder averages of them are generated from $[\log Z(h)]_{\text{av}}$, where $[\cdot]_{\text{av}}$ means to take the average over the disorder. However, direct calculation of $[\log Z(h)]_{\text{av}}$ is formidable challenge in general.

In order to circumvent the difficulty, n copies (replicas) of the system are introduced. Although they have a common disordered environment, each of the replica fields u^a ($a = 1, \dots, n$) couples to independent external sources h^a . Taking the disorder average, the partition function of the replicated system is defined as

$$\mathcal{Z}(\mathbf{h}) := \left[\prod_{a=1}^n Z(h^a) \right]_{\text{av}} = \left[\int \prod_{a=1}^n \mathcal{D}u^a \, e^{\sum_a \beta(-H_{\text{DO}}[u^a] + N h^a \tilde{u}^a)} \right]_{\text{av}},$$

where $\mathbf{h} := (h^1, \dots, h^n)$. Employing $\mathcal{Z}(\mathbf{h})$, the generating function per site $\tilde{w}_N(\mathbf{h})$ is introduced as

$$\tilde{w}_N(\mathbf{h}) := \frac{1}{N\beta} \log \mathcal{Z}(\mathbf{h}) = \frac{1}{N\beta} \log \left[\prod_{a=1}^n Z(h^a) \right]_{\text{av}}. \quad (1)$$

Derivatives of $\tilde{w}_N(\mathbf{h})$ does not directly yield the thermal cumulants averaged over the disorder. It can be transparent using the following notation for the thermal average at $\mathbf{h} = \mathbf{0}$:

$$\langle \cdot \rangle := \frac{1}{Z(0)^n} \int \prod_{a=1}^n \mathcal{D}u_a \cdot e^{-\sum_a \beta H_{\text{Do}}[u^a]}.$$

Namely,

$$\tilde{w}_N(\mathbf{h}) = \frac{1}{N\beta} \log \left[Z(0)^n \left\langle e^{\sum_a N\beta h^a \tilde{u}^a} \right\rangle \right]_{\text{av}}.$$

One finds that $Z(0)^n$ gives non-trivial effect because it depends on the disorder. For instance, the first derivative becomes

$$\partial_a \tilde{w}_N(\mathbf{0}) := \left. \frac{\partial \tilde{w}_N(\mathbf{h})}{\partial h^a} \right|_{\mathbf{h}=\mathbf{0}} = \frac{[Z(0)^n \langle \tilde{u} \rangle]_{\text{av}}}{[Z(0)^n]_{\text{av}}}. \quad (2)$$

A usual way of removing the contribution from $Z(0)^n$ is to take the zero-replica limit $n \rightarrow 0$, which gives $\partial_a \tilde{w}_N(\mathbf{0}) \rightarrow [\langle \tilde{u} \rangle]_{\text{av}}$. It apparently seems that single external source h commonly coupled to all the replicated fields is sufficient for generating the disorder average of the higher cumulants. However, the second derivative at $\mathbf{h} = \mathbf{0}$ becomes

$$\partial_a^2 \tilde{w}_N(\mathbf{0}) \rightarrow N\beta \left([\langle \tilde{u}^2 \rangle]_{\text{av}} - [\langle \tilde{u} \rangle]_{\text{av}}^2 \right)$$

as $n \rightarrow 0$. The result is slightly different from the desired form. For obtaining the correct one, we take the derivative by another source. Namely, for $a \neq b$, we get

$$\partial_a \partial_b \tilde{w}_N(\mathbf{0}) \rightarrow N\beta \left([\langle \tilde{u}^2 \rangle]_{\text{av}} - [\langle \tilde{u} \rangle]_{\text{av}}^2 \right)$$

as $n \rightarrow 0$. Then the second thermal cumulant averaged over the disorder is derived as

$$N\beta \left[\langle \tilde{u}^2 \rangle - \langle \tilde{u} \rangle^2 \right]_{\text{av}} = \lim_{n \rightarrow 0} \left(\partial_a^2 \tilde{w}_N(\mathbf{0}) - \partial_a \partial_b \tilde{w}_N(\mathbf{0}) \right). \quad (3)$$

The above computation demonstrates that we need (at least) two external sources for deriving the disorder average of the second thermal cumulant. It also implies that we need at least p replicas coupled with p independent sources for the disorder average of the p -th thermal cumulant. This fact clearly indicates inconsistency with the zero-replica limit, so that we do not use the limit in the present study.

An alternative way of removing the effect $Z(0)^n$ is to substitute the normalized partition function [18]

$$z(h) := \frac{Z(h)}{Z(0)} \quad (4)$$

for $Z(h)$. Namely, instead of $\tilde{w}_N(\cdot)$ in (1), we adopt $w_N(\cdot)$ defined as the following:

$$w_N(\mathbf{h}) := \frac{1}{N\beta} \log \left[\prod_{a=1}^n z(h^a) \right]_{\text{av}} = \frac{1}{N\beta} \log \left[\left\langle e^{N\beta \sum_{a=1}^n \tilde{u}^a h^a} \right\rangle \right]_{\text{av}}. \quad (5)$$

It is normalized in the sense that $w_N(\mathbf{0}) = 0$. Computation similar to (2) and (3) yields

$$\partial_a w_N(\mathbf{0}) = [\langle \tilde{u} \rangle]_{\text{av}}, \quad (\partial_a^2 - \partial_a \partial_b) w_N(\mathbf{0}) = N\beta \left[\langle \tilde{u}^2 \rangle - \langle \tilde{u} \rangle^2 \right]_{\text{av}} \quad (6)$$

for $a \neq b$.

Now we take the thermodynamic limit

$$w(\mathbf{h}) := \lim_{N \rightarrow \infty} w_N(\mathbf{h}),$$

and define the effective potential $\gamma(\cdot)$ by the Legendre transform:

$$\gamma(\boldsymbol{\varphi}) := \sup_{\mathbf{h}} \left(\sum_{a=1}^n \varphi^a h^a - w(\mathbf{h}) \right). \quad (7)$$

The earlier work of Le Doussal and Wiese showed, with help of the replica trick, that the effective potential of a random elastic model defined from the unnormalized generating function (1) becomes non-analytic in a glass phase if $\varphi^a = \varphi^b$ for $a \neq b$ [15, 16].

Another definition of the effective potential is a free energy with an order parameter fixed. It is referred to as the constraint effective potential (up to an additive constant) [22, 23]. Applying this definition to the replicated system, we first introduce the density function $\rho_N(\cdot)$ as

$$\rho_N(\boldsymbol{\varphi}) := \left[\left\langle \prod_{a=1}^n \delta(\varphi^a - \tilde{u}^a) \right\rangle \right]_{\text{av}}. \quad (8)$$

The constraint effective potential $\hat{\gamma}(\boldsymbol{\varphi})$ is defined as

$$\hat{\gamma}(\boldsymbol{\varphi}) := - \lim_{N \rightarrow \infty} \frac{1}{N\beta} \log \rho_N(\boldsymbol{\varphi}). \quad (9)$$

From (5) and (8), we have

$$e^{N\beta w_N(\mathbf{h})} = \int d\boldsymbol{\varphi} \rho_N(\boldsymbol{\varphi}) e^{N\beta \sum_a \varphi^a h^a},$$

which implies that $\hat{\gamma}(\boldsymbol{\varphi})$ formally satisfies

$$w(\mathbf{h}) = \sup_{\boldsymbol{\varphi}} \left(\sum_a \varphi^a h^a - \hat{\gamma}(\boldsymbol{\varphi}) \right). \quad (10)$$

From (7) and (10), we find that $\gamma(\cdot)$ is the double Legendre transform of $\hat{\gamma}(\cdot)$, which implies that $\gamma(\cdot)$ is the convex hull (envelope) of $\hat{\gamma}(\cdot)$ [23].

The relationship between $\gamma(\cdot)$ and $\hat{\gamma}(\cdot)$ mentioned above is nicely explained in the language of the large deviation principle (LDP) [24, p.23]. According to the literature, $\beta\hat{\gamma}(\cdot)$ is called a rate function. The Legendre transform of it, which is $\beta w(\cdot)$ in the present work, is called the scaled cumulant generating function. The double Legendre transform of the rate function, $\beta\gamma(\cdot)$, is shown to be the convex envelope of $\beta\hat{\gamma}(\cdot)$.

Physical meaning of the effective potential is understood from (9). The probability density for the order parameter can be written as

$$\rho_N(\boldsymbol{\varphi}) \simeq \text{const.} e^{-N\beta\hat{\gamma}(\boldsymbol{\varphi})}$$

for large N . We see that $\boldsymbol{\varphi}$ giving minimum of $\hat{\gamma}(\cdot)$ is realized in the thermodynamic limit. The second thermal cumulant (3) can be computed as

$$N\beta \int d\boldsymbol{\varphi} \left((\varphi^1)^2 - \varphi^1 \varphi^2 \right) \rho_N(\boldsymbol{\varphi}) \simeq \text{const.} \int d\boldsymbol{\varphi} \left((\varphi^1)^2 - \varphi^1 \varphi^2 \right) e^{-N\beta\hat{\gamma}(\boldsymbol{\varphi})}.$$

We see that at least two replicas are needed for the derivation.

In order to understand relationship between a value of the order parameter and form of the effective potential, it is instructive to show a mean-field model for the Ising ferromagnet in pure system. The order parameter φ is the magnetization per site. In the high-temperature phase, the graph of $\hat{\gamma}(\cdot)$ forms like a single well, which has the unique minimum at the origin. It leads to the vanishing order parameter. On the other hand, in the low-temperature phase, the graph of $\hat{\gamma}(\cdot)$ forms a double-well potential symmetric under the Z_2 transform $\varphi \rightarrow -\varphi$. One of the two minima is chosen under a specific boundary condition. Thus a value of the order parameter does not vanish in the low-temperature phase. The other effective potential, $\gamma(\cdot)$, is the convex envelope of $\hat{\gamma}(\cdot)$, whose graph has the flat bottom connecting the two minima of $\hat{\gamma}(\cdot)$. The consequence $\hat{\gamma}(\cdot) \neq \gamma(\cdot)$ originates from the mean-field interaction, where arbitrary two spins are interacting. If the spin interaction is sufficiently short-ranged, we can show that $\hat{\gamma}(\cdot) = \gamma(\cdot)$ [23]. This is because a value of φ can be changed by moving a domain wall just adding boundary energy, which vanishes in the thermodynamic limit.

3. The REM in a magnetic field and its generating function

In this section, we first recall the REM and derive the generating function for its replicated system.

The random energy model (REM) is defined on configurations of N Ising spins $\boldsymbol{\sigma} := \{\sigma_1, \dots, \sigma_N\}$, where every σ_i takes the values of ± 1 [11, 12]. When there is no external field, the energy $E_{\boldsymbol{\sigma}}$ of a spin configuration $\boldsymbol{\sigma}$ is completely independent of how the configuration is. It just follows a Gaussian probability density $P(\cdot)$ specifying disordered environment:

$$P(E_{\boldsymbol{\sigma}}) := \frac{1}{\sqrt{\pi N J^2}} \exp\left(-\frac{E_{\boldsymbol{\sigma}}^2}{N J^2}\right). \quad (11)$$

After magnetic field h is turned on, the energy $E_{\boldsymbol{\sigma}}$ gets dependence on the magnetization $M_{\boldsymbol{\sigma}} := \sum_{i=1}^N \sigma_i$ and is modified to $E_{\boldsymbol{\sigma}} - h M_{\boldsymbol{\sigma}}$. Letting $m_{\boldsymbol{\sigma}}$ be the magnetization per site $M_{\boldsymbol{\sigma}}/N$, the partition function becomes

$$Z(h) := \sum_{\boldsymbol{\sigma}} e^{-\beta E_{\boldsymbol{\sigma}} + \beta N m_{\boldsymbol{\sigma}} h}. \quad (12)$$

Now we compute the generating function for the replicated system of the REM. As we stressed in the previous section, we use the normalized generating function $w_N(\mathbf{h})$ defined by (5) instead of $\tilde{w}_N(\mathbf{h})$ plus the replica trick. As a by-product, we do not need the free parameter n , so that we can investigate the simplest but non-trivial case, $n = 2$. Namely, we deal with

$$w_N(h^1, h^2) := \frac{1}{N\beta} \log \left[z(h^1) z(h^2) \right]_{\text{av}}, \quad (13)$$

where $z(\cdot)$ is defined in (4) with use of (12). It is easily checked in the same way as for (6) that $w_N(h^1, h^2)$ actually generates the disorder average of the second cumulants for

m_{σ} by the following formula:

$$\left[\langle m_{\sigma}^2 \rangle - \langle m_{\sigma} \rangle^2 \right]_{\text{av}} = \frac{1}{N\beta} \left(\partial_1^2 w_N(0,0) - \partial_1 \partial_2 w_N(0,0) \right). \quad (14)$$

In order to compute the right-hand side of (13), let us derive a general formula to $[\langle O(\sigma^1, \sigma^2) \rangle]_{\text{av}}$, where $O(\sigma^1, \sigma^2)$ depends on the replicated spin configurations, but not on the quenched random variables $\{E_{\sigma}\}$. The thermal average for the two replicas is

$$\langle O(\sigma^1, \sigma^2) \rangle := \frac{1}{Z(0)^2} \sum_{\sigma^1, \sigma^2} O(\sigma^1, \sigma^2) e^{-\beta E_{\sigma^1} - \beta E_{\sigma^2}}.$$

Since each of E_{σ} independently follows (11), we split the summation into the two cases, $\sigma^1 = \sigma^2$ and $\sigma^1 \neq \sigma^2$, when we take the disorder average. Thus we have

$$\begin{aligned} [\langle O(\sigma^1, \sigma^2) \rangle]_{\text{av}} &= \left[\frac{1}{Z(0)^2} e^{-2\beta E_{\sigma}} \right]_{\text{av}} \sum_{\sigma} O(\sigma, \sigma) \\ &+ \left[\frac{1}{Z(0)^2} e^{-\beta E_{\sigma^1} - \beta E_{\sigma^2}} \right]_{\text{av}} \sum_{\sigma^1 \neq \sigma^2} O(\sigma^1, \sigma^2). \end{aligned} \quad (15)$$

The first factor is written as

$$\left[\frac{1}{Z(0)^2} e^{-2\beta E_{\sigma}} \right]_{\text{av}} = 2^{-N} \left[\frac{1}{Z(0)^2} \sum_{\sigma} e^{-2\beta E_{\sigma}} \right]_{\text{av}} = 2^{-N} [Y_N]_{\text{av}}, \quad (16)$$

where

$$Y_N := \frac{1}{Z(0)^2} \sum_{\sigma} e^{-2\beta E_{\sigma}}$$

is known as the participation ratio [25, p.100]. The second factor is also expressed using $[Y_N]_{\text{av}}$ as

$$\left[\frac{1}{Z(0)^2} e^{-\beta E_{\sigma^1} - \beta E_{\sigma^2}} \right]_{\text{av}} = \frac{1}{2^N(2^N - 1)} \left[\frac{1}{Z(0)^2} \sum_{\sigma^1 \neq \sigma^2} e^{-\beta E_{\sigma^1} - \beta E_{\sigma^2}} \right]_{\text{av}} = \frac{1 - [Y_N]_{\text{av}}}{2^N(2^N - 1)}. \quad (17)$$

Insertion of (16) and (17) to (15) leads to

$$[\langle O(\sigma^1, \sigma^2) \rangle]_{\text{av}} = p_N 2^{-N} \sum_{\sigma} O(\sigma, \sigma) + (1 - p_N) 2^{-2N} \sum_{\sigma^1, \sigma^2} O(\sigma^1, \sigma^2). \quad (18)$$

Here we have used the notation

$$p_N := \frac{[Y_N]_{\text{av}} - 2^{-N}}{1 - 2^{-N}},$$

which obviously has the same thermodynamic limit as $[Y_N]_{\text{av}}$. According to the literature [25, p.101, p.153] (see also [26]),

$$\lim_{N \rightarrow \infty} p_N = \lim_{N \rightarrow \infty} [Y_N]_{\text{av}} = \begin{cases} 0 & (\beta < \beta_c) \\ 1 - \frac{\beta_c}{\beta} & (\beta \geq \beta_c) \end{cases}, \quad (19)$$

where $\beta_c := 2\sqrt{\log 2}/J$ is the critical temperature dividing the paramagnetic phase ($\beta < \beta_c$) and the glass phase ($\beta > \beta_c$) in the REM [11].

Now we turn back to (13). Using the explicit form of $z(\cdot)$, we have

$$\left[z(h^1) z(h^2) \right]_{\text{av}} = \left[\left\langle e^{\beta N(h^1 m_{\sigma^1} + h^2 m_{\sigma^2})} \right\rangle \right]_{\text{av}}. \quad (20)$$

The right-hand side is simply evaluated letting $O(\sigma^1, \sigma^2) = \exp(\beta N(h^1 m_{\sigma^1} + h^2 m_{\sigma^2}))$ in (18). Since the exponent is regarded as a Hamiltonian of non-interacting Ising spins in a uniform magnetic field, we find that the right-hand side of (20) is written as

$$\left[z(h^1) z(h^2) \right]_{\text{av}} = A_N + B_N \quad (21)$$

with

$$\begin{aligned} A_N &:= p_N \left(\text{ch} \left(\beta (h^1 + h^2) \right) \right)^N \\ B_N &:= (1 - p_N) \left(\text{ch}(\beta h^1) \text{ch}(\beta h^2) \right)^N. \end{aligned} \quad (22)$$

Let us take the thermodynamic limit of $w_N(h^1, h^2)$.

$$w(h^1, h^2) := \lim_{N \rightarrow \infty} w_N(h^1, h^2) = \lim_{N \rightarrow \infty} \frac{1}{N\beta} \log(A_N + B_N). \quad (23)$$

When $\beta < \beta_c$, $p_N \rightarrow 0$ as $N \rightarrow \infty$ according to (19), so that A_N vanishes in the thermodynamic limit. Thus we get

$$w(h^1, h^2) = \frac{1}{\beta} \left(\log \text{ch}(\beta h^1) + \log \text{ch}(\beta h^2) \right). \quad (24)$$

Namely $w(h^1, h^2)$ is analytic on the whole $h^1 h^2$ plane in the high-temperature phase.

On the other hand, when $\beta \geq \beta_c$, we need to find which exponentially dominates A_N or B_N for large N . It is readily determined if we notice that $\text{ch}(a+b) = \text{ch} a \text{ch} b + \text{sh} a \text{sh} b$. The result is

$$w(h^1, h^2) = \begin{cases} \frac{1}{\beta} \log \text{ch}(\beta(h^1 + h^2)) & (h^1 h^2 \geq 0) \\ \frac{1}{\beta} (\log \text{ch}(\beta h^1) + \log \text{ch}(\beta h^2)) & (h^1 h^2 < 0) \end{cases}. \quad (25)$$

It is continuous on the whole $h^1 h^2$ plane but not differentiable on the lines $h^1 = 0$ and $h^2 = 0$. In fact,

$$\partial_a w(h^1, h^2) = \begin{cases} \text{th}(\beta(h^1 + h^2)) & (h^1 h^2 > 0) \\ \text{th}(\beta h^a) & (h^1 h^2 < 0) \end{cases} \quad (26)$$

for $a = 1, 2$. It indicates that $\partial_a w(h^1, h^2)$ is not continuous on $h^a = 0$. For example, when $h > 0$, we get

$$\begin{aligned} \lim_{h^2 \uparrow 0} \partial_2 w(h, h^2) &= 0, \\ \lim_{h^2 \downarrow 0} \partial_2 w(h, h^2) &= \text{th}(\beta h) \neq 0. \end{aligned} \quad (27)$$

This non-analytic behaviour, which is depicted in Fig.1, plays an crucial role to differentiability of the effective potential.

Note that we cannot apply the formula (14) after taking the thermodynamic limit since the partial derivatives do not exist on the lines $h^a = 0$ ($a = 1, 2$). For finite N , straightforward calculation gives

$$\left(\partial_a^2 - \partial_a \partial_b \right) w_N(h^1, h^2) = \frac{\beta B_N}{c_a^2 (A_N + B_N)} + \frac{N \beta A_N B_N}{(A_N + B_N)^2} (t_a - t_b)(t_a - t_{12}). \quad (28)$$

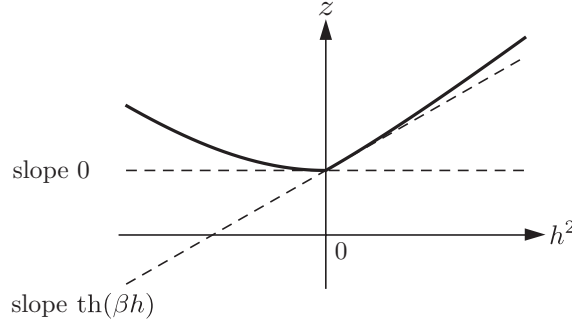


Figure 1: The solid curve is the graph of $z = w(h, h^2)$ with fixed $h > 0$. The dashed lines represent tangential lines at $h^2 = 0$. The dashed lines with slopes 0 and $\text{th}(\beta h)$ are respectively the left and the right derivative at $h^2 = 0$.

where $(a, b) = (1, 2)$ or $(a, b) = (2, 1)$, and we have used the following abbreviation: $c_a := \text{ch}(\beta h^a)$, $t_a := \text{th}(\beta h^a)$, $t_{12} := \text{th}(\beta(h^1 + h^2))$. Letting $h^1 = h^2 = 0$, we see from (14), (22) and (28) that the susceptibility χ_N is computed as

$$\chi_N := N\beta \left[\langle m_{\sigma}^2 \rangle - \langle m_{\sigma} \rangle^2 \right]_{\text{av}} = (\partial_a^2 - \partial_a \partial_b) w_N(0, 0) = \beta(1 - p_N).$$

Using (19), we have the thermodynamic limit.

$$\lim_{N \rightarrow \infty} \chi_N = \begin{cases} \beta & (\beta < \beta_c) \\ \beta_c & (\beta \geq \beta_c) \end{cases}, \quad (29)$$

which is precisely equal to the susceptibility first obtained by Derrida [11, 12], as expected. Note that it holds for both $(a, b) = (1, 2)$ and $(a, b) = (2, 1)$, which reflects that the replica symmetry is preserved in the finite system when $(h^1, h^2) = (0, 0)$.

The same result is obtained by the following limiting procedure with the explicit replica-symmetry breaking by $(h^1, h^2) = (h, 0)$. If $a = 1, b = 2$, we get

$$\lim_{h \rightarrow 0} \lim_{N \rightarrow \infty} (\partial_1^2 - \partial_1 \partial_2) w_N(h, 0) = \begin{cases} \beta & (\beta < \beta_c) \\ \beta_c & (\beta \geq \beta_c) \end{cases}$$

according to (28). In this formula, however, the replica indices are *no longer* exchangeable. In fact, if $a = 2, b = 1$

$$\lim_{h \rightarrow 0} \lim_{N \rightarrow \infty} (\partial_2^2 - \partial_2 \partial_1) w_N(h, 0) = \begin{cases} \beta & (\beta < \beta_c) \\ \infty & (\beta \geq \beta_c) \end{cases}.$$

The infinity originates from the second term in (28) proportional to N . if $a = 1, b = 2$, it vanishes because $t_{12} - t_a = 0$ in this term, while it remains if $a = 2, b = 1$. It can be interpreted as the symmetry by permutation of the replica indices is spontaneously broken. Namely the spontaneous RSB with the original meaning takes place. A similar observation is performed in [26], where an inter-replica couplings are introduced in the Hamiltonian as symmetry breaking terms.

4. The Effective Potential

In this section, we derive the effective potential $\gamma(\varphi^1, \varphi^2)$ by the Legendre transform (7) of $w(h^1, h^2)$. Here, if $w(h^1, h^2)$ is differentiable, the Legendre transform can be carried out by solving the following equations

$$\varphi^a = \partial_a w(h^1, h^2), \quad (a = 1, 2) \quad (30)$$

for h^1 and h^2 , and then inserting the solutions into the right-hand side of (7). We can easily derive $\gamma(\varphi^1, \varphi^2)$ in the high-temperature phase along this line. In fact, from (24), the equations (30) become

$$\varphi^a = \text{th}(\beta h^a), \quad (a = 1, 2) \quad (31)$$

for all h^1 and h^2 . Solving them for h^1 and h^2 , we obtain

$$\gamma(\varphi^1, \varphi^2) = -\frac{1}{\beta} (s(\varphi^1) + s(\varphi^2)), \quad (32)$$

where

$$s(\varphi) := -\frac{1}{2} ((1 + \varphi) \log(1 + \varphi) + (1 - \varphi) \log(1 - \varphi)).$$

It indicates that $\gamma(\varphi^1, \varphi^2)$ has the global minimum at the origin and has no singularity.

In the low-temperature phase, first we consider the case of $h^1 h^2 > 0$. We use the first line of (25) for (30), which yields

$$\varphi^a = \text{th} \beta (h^1 + h^2), \quad (a = 1, 2).$$

It shows that the identity $\varphi^1 = \varphi^2$ holds. Inserting the solution for $h^1 + h^2$ to (7) yields

$$\varphi^1 h^1 + \varphi^2 h^2 - w(h^1, h^2) = \varphi^1 (h^1 + h^2) - \frac{1}{\beta} \log \text{ch} \beta (h^1 + h^2) = -\frac{1}{\beta} s(\varphi^1).$$

Note that this is the effective potential in the case of $\varphi^1 = \varphi^2$.

Next we go to the case of $h^1 h^2 < 0$. Since $w(\cdot, \cdot)$ is given by the second line of (25), the result is same as the case of the high-temperature phase (32). According to (31), the condition $h^1 h^2 < 0$ is translated to $\varphi^1 \varphi^2 < 0$. Thus the results for $h^1 h^2 > 0$ and for $h^1 h^2 < 0$ are summarized as

$$\gamma(\varphi^1, \varphi^2) = \begin{cases} -\frac{1}{\beta} s(\varphi^1), & (\varphi^1 = \varphi^2) \\ -\frac{1}{\beta} (s(\varphi^1) + s(\varphi^2)) & (\varphi^1 \varphi^2 < 0) \end{cases} \quad (33)$$

In order to determine $\gamma(\varphi^1, \varphi^2)$ for all φ^1 and φ^2 ($|\varphi^a| < 1$, $a = 1, 2$), we have to investigate the case of $h^1 h^2 = 0$. In this case, a partial derivative does not exist as we have seen in the previous section, so that we employ the following geometric interpretation of the Legendre transform (7): for a given φ^1 and φ^2 , consider the plane defined by the formula

$$z = \varphi^1 h^1 + \varphi^2 h^2 + z_0 \quad (34)$$

in the $h^1 h^2 z$ space. We choose z_0 in such a way that the plane has a common point with the surface $z = w(h^1, h^2)$ and try to minimize the value of z_0 , then the minimum value equals $-\gamma(\varphi^1, \varphi^2)$.

In order to find the minimum of z_0 when $h^1 h^2 = 0$, we first consider the case of $h^2 = 0$ and $h^1 > 0$. Take an arbitrary point $(h, 0)$ with $h > 0$ and let the corresponding point on the surface $z = w(h^1, h^2)$ be $P(h, 0, w(h, 0))$. We choose φ^1 , φ^2 and z_0 in such a way that the plane (34) contact with the surface $z = w(h^1, h^2)$ at P. Since $\partial_1 w(h, 0)$ is well-defined according to (25), φ^1 is uniquely determined as

$$\varphi^1 = \partial_1 w(h, 0) = \text{th}(\beta h). \quad (35)$$

On the other hand, $\partial_2 w(h, 0)$ does not exist as we have seen in (27). In this case, φ^2 can take the value between the left and the right derivatives, hence $\varphi^2 \in [0, \text{th}(\beta h)] = [0, \varphi^1]$. Since the point P is on the plane (34), we find that $z_0 = w(h, 0) - \varphi^1 h = s(\varphi^1)/\beta$. See Fig 2. Note that if z_0 took a value less than $s(\varphi^1)/\beta$, the plane (34) would not have a common point with the surface. It indicates that $s(\varphi^1)/\beta$ gives the minimum. We thus have

$$\gamma(\varphi^1, \varphi^2) = -s(\varphi^1)/\beta \quad (36)$$

for $\varphi^2 \in [0, \varphi^1]$. Similar calculation can be applied in the case when $h^2 = 0$, $h^1 < 0$ and we obtain (36) for $\varphi^2 \in [\varphi^1, 0]$.

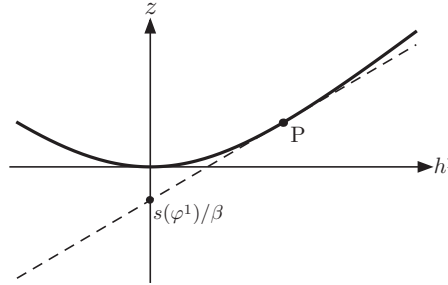


Figure 2: The sectional plane $h^2 = 0$ in the $h^1 h^2 z$ space. The solid line is the cross section of the surface $z = w(h^1, h^2)$. The dashed line represents the plane $z = \varphi^1 h^1 + \varphi^2 h^2 + z_0$ contacting with the surface at $P(h, 0, w(h, 0))$. It intercepts the z axes at $s(\varphi^1)/\beta$, which is equal to $-\gamma(\varphi^1, \varphi^2)$.

When $h^1 = 0$, exchanging the role of φ^1 and φ^2 in the case of $h^2 = 0$, we get

$$\gamma(\varphi^1, \varphi^2) = -s(\varphi^2)/\beta \quad (37)$$

for $\varphi^1 \in [0, \varphi^2]$ or $\varphi^1 \in [\varphi^2, 0]$. Combining the results (33) (36) and (37), we finally obtain

$$\gamma(\varphi^1, \varphi^2) = \begin{cases} -\frac{1}{\beta}s(\varphi^1) & (0 \leq \varphi^2 \leq \varphi^1 \text{ or } \varphi^1 \leq \varphi^2 \leq 0) \\ -\frac{1}{\beta}s(\varphi^2) & (0 \leq \varphi^1 \leq \varphi^2 \text{ or } \varphi^2 \leq \varphi^1 \leq 0) \\ -\frac{1}{\beta}(s(\varphi^1) + s(\varphi^2)) & (\varphi^1 \varphi^2 < 0) \end{cases} \quad (38)$$

As is shown in Fig.3, regions that specify the values of $\gamma(\varphi^1, \varphi^2)$ have the boundaries $\varphi^a = 0$ ($a = 1, 2$) and $\varphi^2 = \varphi^1$, on which it is continuous but non-analytic.

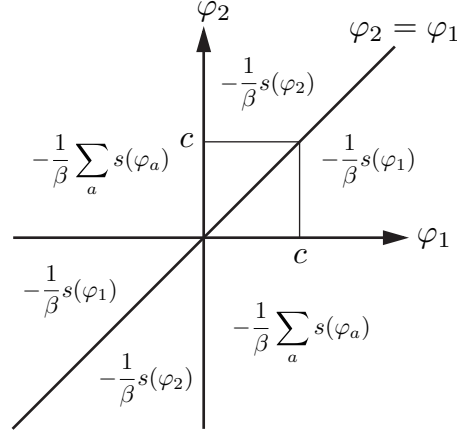


Figure 3: Values of $\gamma(\varphi^1, \varphi^2)$ on the $\varphi^1 \varphi^2$ plane. The segments on $\varphi^1 = c$ and $\varphi^2 = c$ show contours with the value $\gamma(\varphi^1, \varphi^2) = -\frac{1}{\beta} s(c)$. They meet at $\varphi^1 = \varphi^2 = c$, where the effective potential becomes non-analytic.

The non-analyticity on $\varphi^1 = \varphi^2$ is observed in fixed-point potentials of the Functional renormalization group transformation in various disordered systems having short-range interaction [15–19, 21, 27, 28]. Following [21], it is convenient to introduce the variables $x := (\varphi^1 + \varphi^2)/2$ and $y := (\varphi^1 - \varphi^2)/2$. For fixed $x > 0$ and for small y satisfying $|y| < x$, the effective potential is written as

$$\gamma(\varphi^1, \varphi^2) = -\frac{1}{\beta} s(x + |y|). \quad (39)$$

We see the linear cusp at $y = 0$, which resembles the non-analytic effective potential in random $O(N)$ models studied in [15, 16, 21]. It should be noted that the order of the singularity is different. In fact, in the $O(N)$ model, the linear cusp $|y|$ appears in the second derivatives of the effective potential with respect to φ^1 and φ^2 .

5. The constraint effective potential

In this section we exactly calculate the constraint effective potential for two-replica system of the REM. Here the density function defined in (8) needs slight modification in accordance with discrete spin variables, i.e.,

$$\rho_N(\varphi^1, \varphi^2) := \left[\left\langle \delta(\varphi^1, m_{\sigma^1}) \delta(\varphi^2, m_{\sigma^2}) \right\rangle \right]_{\text{av}},$$

where $\delta(x, y) = 1$ if $x = y$ and $\delta(x, y) = 0$ otherwise. The constraint effective potential $\hat{\gamma}(\varphi^1, \varphi^2)$ is defined as in (9):

$$\hat{\gamma}(\varphi^1, \varphi^2) := -\lim_{N \rightarrow \infty} \frac{1}{N\beta} \log \rho_N(\varphi^1, \varphi^2).$$

In the present study, we easily calculate $\rho_N(\cdot, \cdot)$ employing (18).

$$\rho_N(\varphi^1, \varphi^2) = p_N 2^{-N} n(N\varphi^1) \delta(\varphi^1, \varphi^2) + (1 - p_N) 2^{-2N} n(N\varphi^1) n(N\varphi^2), \quad (40)$$

where

$$n(M) := \binom{N}{\frac{N+M}{2}}$$

is the number of configurations that have the total magnetization M . Employing the Stirring formula, we have

$$\rho_N(\varphi^1, \varphi^2) \simeq p_N e^{Ns(\varphi^1)} \delta(\varphi^1, \varphi^2) + (1 - p_N) e^{N(s(\varphi^1) + s(\varphi^2))} \quad (41)$$

for large N . When $\beta \leq \beta_c$, since $p_N \simeq 0$ from (19), we get

$$\hat{\gamma}(\varphi^1, \varphi^2) = -\frac{1}{\beta} (s(\varphi^1) + s(\varphi^2)),$$

which is identical with $\gamma(\varphi^1, \varphi^2)$ obtained in (32). When $\beta > \beta_c$, we find that the first term in (41) dominates on the line $\varphi^1 = \varphi^2$, thus we conclude that

$$\hat{\gamma}(\varphi^1, \varphi^2) = \begin{cases} -\frac{1}{\beta} s(\varphi^1) & (\varphi^1 = \varphi^2) \\ -\frac{1}{\beta} (s(\varphi^1) + s(\varphi^2)) & (\text{otherwise}) \end{cases}. \quad (42)$$

The difference between (38) and (42) can be understood from a general argument in section 2. Namely, $\gamma(\cdot, \cdot)$ is the convex hull (envelope) of $\hat{\gamma}(\cdot, \cdot)$.

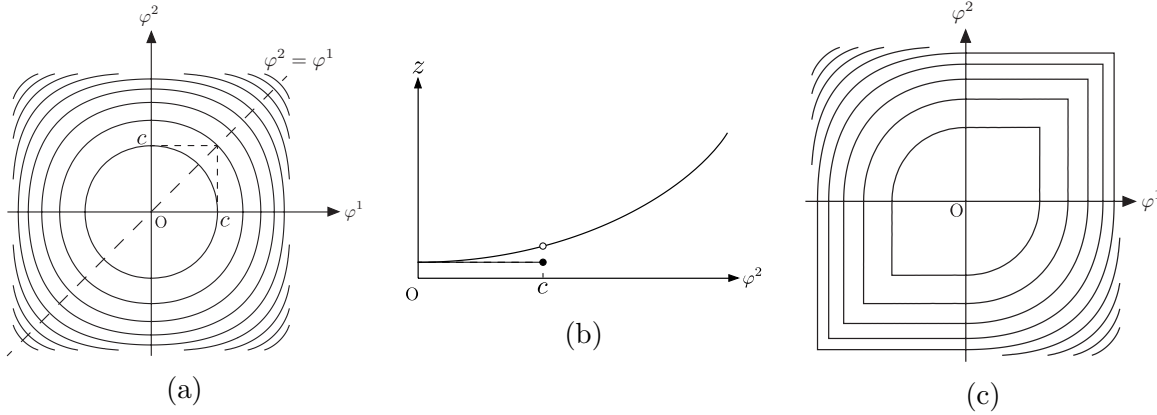


Figure 4: Graphs for $\hat{\gamma}(\varphi^1, \varphi^2)$ and for its convex envelope. (a) Contours for the surface $z = \hat{\gamma}(\varphi^1, \varphi^2)$. (b) The graph of $z = \hat{\gamma}(c, \varphi^2)$ for fixed c . (c) Contours for the convex envelope of $z = \hat{\gamma}(\varphi^1, \varphi^2)$

This fact can be confirmed by the following argument: figure 4a shows contours for the graph $z = \hat{\gamma}(\varphi^1, \varphi^2)$. It has the minimum at the origin. It should be noted that it is discontinuous on $\varphi^2 = \varphi^1$, thus the solid curves are not applicable on the line $\varphi^2 = \varphi^1$. The explicit formula (42) indicates that

$$\hat{\gamma}(c, c) = \hat{\gamma}(0, c) = \hat{\gamma}(c, 0).$$

Therefore, the section of the surface by $\varphi^1 = c$ becomes as shown in figure 4b. We also have the same curve for the section by $\varphi^2 = c$. It implies that we can make the convex hull of $\hat{\gamma}(\cdot, \cdot)$ by connecting $(c, c, \hat{\gamma}(c, c))$ to $(c, 0, \hat{\gamma}(c, 0))$, and to $(0, c, \hat{\gamma}(0, c))$ with the horizontal segments. The resultant surface has the contours in figure 4c. It coincides with contours of $\gamma(\cdot, \cdot)$. See figure 3.

From the view point of the construction of $\gamma(\cdot, \cdot)$ from $\hat{\gamma}(\cdot, \cdot)$, we can conclude that the non-analyticity in $\gamma(\cdot, \cdot)$ results from the discontinuity of $\hat{\gamma}(\cdot, \cdot)$.

6. Summary and Discussion

In this paper, we have exactly derived the effective potentials of the two-replica system consisting of the REM following the two definitions (7) and (9). It is found that $\gamma(\varphi^1, \varphi^2)$, which is defined by (7), is continuous but non-analytic on $\varphi^1 = \varphi^2$ in the low-temperature phase. The result is similar to the effective potential in $O(N)$ models studied in [15, 16, 21] although the order of the singularity is different. The other effective potential $\hat{\gamma}(\varphi^1, \varphi^2)$ defined by (9) is discontinuous on the line $\varphi^1 = \varphi^2$. The potential surface on this line becomes lower than vicinity and has a gap. Since $\gamma(\cdot, \cdot)$ is the convex envelope of $\hat{\gamma}(\cdot, \cdot)$, we can interpret that the non-analyticity of $\gamma(\cdot, \cdot)$ is caused by the discontinuity appearing in $\hat{\gamma}(\cdot, \cdot)$.

In order to see the origin of the discontinuity in detail, let us consider the probability density of the replica overlap in the REM

$$P_N(q) := \left[\left\langle \delta \left(q, \frac{1}{N} \sum_{i=1}^N \sigma_i^1 \sigma_i^2 \right) \right\rangle \right]_{\text{av}}.$$

The right-hand side is evaluated using (18) as

$$P_N(q) \simeq p_N \delta(q, 1) + (1 - p_N) e^{Ns(q)}$$

for large N . In the low-temperature phase, we have the well-known thermodynamic limit (e.g., [25, p.162], [29, p.180], [30])

$$\lim_{N \rightarrow \infty} P_N(q) = \left(1 - \frac{\beta_c}{\beta} \right) \delta(q, 1) + \frac{\beta_c}{\beta} \delta(q, 0).$$

Thus, when we pick out the two states following the Boltzmann measure, the probability for the two states to become identical each other is non-negligible. It happens because a smaller-than-exponential set of configurations dominates the Boltzmann measure, which is referred to as the condensation phenomenon [25, p.100]. It causes the discontinuous gap of the surface $z = \hat{\gamma}(\varphi^1, \varphi^2)$ along $\varphi^1 = \varphi^2$. Consequently, $\gamma(\varphi^1, \varphi^2)$, the double Legendre transform of $\hat{\gamma}(\varphi^1, \varphi^2)$, becomes non-analytic on $\varphi^1 = \varphi^2$.

Since the condensation phenomenon is considered as a typical feature of a glass phase in mean-field models, it is plausible that non-analytic effective potential appears together with the RSB as far as mean-field models are concerned. However, it is unclear

whether the same mechanism takes place in short-ranged disordered models. In fact, the non-convexity of $\hat{\gamma}(\cdot, \cdot)$ will strongly depend on mean-field property of the REM, while thermodynamic stability in short-ranged models ensures convexity of a constraint effective potential [23]. Further investigation will shed light on universal relationship between the non-analyticity and the RSB.

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